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Spherical Ornstein-Uhlenbeck processes

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Abstract The paper considers random motion of a point on the surface of a sphere, in the case where the angular velocity is determined by an Ornstein-Uhlenbeck process. The solution is fully characterized by only one dimensionless number, the persistence angle, which is the typical angle of rotation of the object during the correlation time of the angular velocity.

We first show that the two-dimensional case is exactly solvable. When the persistence angle is large, a series for the correlation function has the surprising property that its sum varies much more slowly than any of its individual terms.

In three dimensions we obtain asymptotic forms for the correlation function, in the limits where the persistence angle is very small and very large. The latter case exhibits a complicated transient, followed by a much slower exponential decay. The decay rate is determined by the solution of a radial Schrödinger equation in which the angular momentum quantum number takes an irrational value, namely $j = \frac{1}{2}(\sqrt{17} - 1)$.

Possible applications of the model to objects tumbling in a turbulent environment are discussed.

Keywords Diffusion, Ornstein-Uhlenbeck process

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1 Introduction

There are many contexts in which random motion is confined to the surface of a sphere. Examples include the motion of a unit vector $\mathbf{n}(t)$ indicating the orientation of an object tumbling in a turbulent fluid flow, or the advection of a tracer in a thin, turbulent planetary atmosphere. In some applications it is sufficient to model the motion as diffusion on the surface of a sphere, which can be solved by noting that the eigenfunctions of the diffusion operator are spherical harmonics. In other applications, however, the angular velocity varies smoothly as a function of time, and the diffusive approximation

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is not valid. Our objective is to obtain insight into random motion of a unit vector, by studying what is perhaps the simplest model.

Smooth random motion on a sphere can be characterised by a dimensionless parameter which measures the typical angular distance through which the point has turned in the timescale for relaxation of fluctuations of the angular velocity. Due to an analogy with the concept of persistence length in polymer physics [1], we term this parameter the *persistence angle*: it will be denoted by β . It is desirable to have a model which is a physically well motivated description of some processes, in which β appears as a parameter. This paper discusses such a model, which is an extension of the well-known Ornstein-Uhlenbeck process [2,3] to describe motion on a circle or a sphere.

We start by describing the standard Ornstein-Uhlenbeck equation on a line, in order to introduce some notation and elementary ideas. The Ornstein-Uhlenbeck process is a stochastic differential equation for the time-dependence of a variable $v(t)$:

$$\dot{v} = -\gamma v + \sqrt{2D}\eta(t) \quad (1)$$

where $\eta(t)$ is a white-noise signal, satisfying

$$\langle \eta(t) \rangle = 0, \quad \langle \eta(t)\eta(t') \rangle = \delta(t-t'). \quad (2)$$

Throughout this paper, $\langle X \rangle$ is the expectation value of X . The process equilibrates to a statistically stationary state, characterised by the following correlation function:

$$\langle v(t+\Delta t)v(t) \rangle = \frac{D}{\gamma} \exp(-\gamma|\Delta t|). \quad (3)$$

Equation (1) may be considered purely as a model for the fluctuations of velocity v of a particle, or it may be combined with the equation $\dot{x} = v$ to give a model for the displacement x . The motion in space is ballistic when viewed on short timescales, but on long timescales it is diffusive, with the displacement Δx satisfying $\langle \Delta x \rangle = 0$ and $\langle \Delta x^2 \rangle \sim 2Dt$, with spatial diffusion constant D :

$$D = \frac{1}{2} \int_{-\infty}^{\infty} dt \langle v(t)v(0) \rangle = \frac{D}{\gamma^2}. \quad (4)$$

To generalize the Ornstein-Uhlenbeck model on a line to a circle, we simply replace the variable v by the angular velocity, ω , and the displacement along the line, x , by the angle along the circle, θ . In two dimensions, we are considering the Ornstein-Uhlenbeck process on a manifold which has the simplest closed topology, namely a circle.

If the equation of motion (1) is interpreted as a description of an angular velocity ω , then D has dimension $[D] = \text{T}^{-3}$. The damping rate has dimension $[\gamma] = \text{T}^{-1}$. Thus, the problem is completely characterized by one single dimensionless parameter constructed from D and γ ; we take this to be

$$\beta = \sqrt{\frac{D}{\gamma^3}}. \quad (5)$$

The parameter β has the following simple physical interpretation. The typical angular velocity is, according to (3), $\sqrt{D/\gamma}$, and its fluctuations occur on a timescale γ^{-1} . The typical angle of rotation over the correlation timescale of the angular velocity is then $\Delta\theta \sim \sqrt{D/\gamma}/\gamma = \beta$, so that β does correspond to a persistence angle.

In this paper we show how to compute statistics characterising both circular and spherical Ornstein-Uhlenbeck processes, such as the correlation function

$$C(t) = \langle \mathbf{n}(t) \cdot \mathbf{n}(0) \rangle. \quad (6)$$

The solution to this problem has very different properties in two and three dimensions.

In two dimensions, exact formulae are obtained for $C(t)$ by expressing this correlation function in terms of the eigenvalues and eigenfunctions of the Fokker-Planck equation describing the probability

density of the angular velocity and angle variables. We find that the correlation function $C(t)$ can be expressed as a series:

$$C(t) = \exp(-\beta^2 \gamma t) \sum_{N=0}^{\infty} C_N(\beta) \exp(-N \gamma t) \quad (7)$$

where the coefficients $C_N(\beta)$ are determined explicitly, and satisfy $C_N(0) = \delta_{N,0}$. We show that series given by Eq.7 can be summed exactly, which reduces the correlation function to a very simple analytic form. There is, however, a surprising feature of this series expansion which deserves comment. Although the series (7) is convergent for all t and for all β , in the limit as $\beta \rightarrow \infty$ the structure of (7) appears to be hard to reconcile with physical expectations. We expect that when $\beta \gg 1$, the object rotates with almost constant angular velocity, and the decay of correlations results from motions with different angular velocity getting out of phase. This occurs on a timescale $\sqrt{\gamma/D} = 1/(\gamma\beta)$, larger than the timescale for decay of the leading factor in (7), which is $1/(\gamma\beta^2)$. The implication is that, when $\beta \gg 1$, the summation over the exponentially *decreasing* terms approaches an *increasing* exponential, which almost cancels the rapid decay of $\exp(-\beta^2 \gamma t)$. We show how this behaviour is realised, by taking very large coefficients with alternating signs. This effect is analogous to a phenomenon known as superoscillation [4], where a Fourier sum oscillates faster than appears to be possible due to its bandwidth. In the case where $\beta \ll 1$, the motion of $\mathbf{n}(t)$ is well approximated by diffusion on the circle, and in that limit (7) yields $C(t) = \exp(-Dt/\gamma^2)$, as expected.

The extension of the problem to three-dimensions requires the introduction of a *vector* to describe the angular velocity. In addition, the closed manifold that describes the configuration of the system has a significantly more complicated topology. It turns out that in three-dimensions, the Ornstein-Uhlenbeck model of dynamic on a sphere does not appear to allow an exact solution, and we show that the spectrum has a very different structure from that of the two-dimensional case. When $\beta \rightarrow 0$, the dynamics is easily understood in terms of diffusion on the surface of a sphere, for which the eigenfunctions are spherical harmonics. The solution has a much more complex behaviour in the limit as $\beta \rightarrow \infty$. As for the two-dimensional case, there is a transient which occurs on a timescale $1/\gamma\beta$, but this transient is followed by a slow exponential decay of the correlation function on a timescale $\sim 1/\gamma$. We show that in the limit as $\beta \rightarrow \infty$ the slowest-decaying mode contributing to the correlation function is obtained from a radial Schrödinger equation for which the quantum number j takes an *irrational* value. Physically, the two time-scale solution we find for the correlation function can be explained by noting that when the angular velocity is constant, the component of \mathbf{n} parallel to ω is also a constant. The existence of this invariance explains the slow decorrelation of \mathbf{n} at long times.

Because this problem is far from straightforward we start by considering the two-dimensional case, where the Ornstein-Uhlenbeck process has a circular coordinate. In section 2 the correlation function (6) is determined in closed form for the circular Ornstein-Uhlenbeck process. The spherical Ornstein-Uhlenbeck process is discussed in section 3, where we show that the Fokker-Planck operator can be transformed into a quantum Hamiltonian for a spin coupled to a spherical harmonic oscillator. Section 3 also considers the symmetry properties of the Fokker-Planck operator and the use of the eigenfunctions of the three-dimensional harmonic oscillator as a convenient basis set. The asymptotic behaviour of the solution for both large and small β is considered in section 4. Section 5 contains some discussion of possible areas of application of the model. Technical details of matrix elements required in section 4 are discussed in appendix A. Appendix B discusses an alternative model for continuous random motion on a circle, which is required to support the discussion in section 5.

2 Two-dimensional case

2.1 Formulation and general solution

We consider motion on a circle, with angular coordinate θ , angular velocity ω , replacing equation (1) by:

$$\dot{\theta} = \omega, \quad \dot{\omega} = -\gamma\omega + \sqrt{2D}\eta(t) \quad (8)$$

where $\eta(t)$ is a standard white noise signal, with statistics satisfying (2), and where γ , D are the damping and diffusion constants. Our aim is to be able to compute correlation functions such as

$\langle \mathbf{n}(t) \cdot \mathbf{n}(0) \rangle$, where $\mathbf{n}(t)$ is a unit vector with direction θ . This correlation function is obtained by computing $\langle \cos \theta(t) \rangle$ subject to the initial condition $\theta(0) = 0$, with ω having its equilibrium distribution.

Statistics of the model (8) are obtained by computing the joint probability density of θ and ω , $P(\theta, \omega, t)$, with an appropriate initial condition. This probability density satisfies a Fokker-Planck equation [3]

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial \theta}(\omega P) + \gamma \frac{\partial}{\partial \omega}(\omega P) + D \frac{\partial^2 P}{\partial \omega^2} \equiv \hat{\mathcal{F}}P. \quad (9)$$

This equation is solved by determining the spectrum and eigenfunctions of the Fokker-Planck operator $\hat{\mathcal{F}}$, satisfying $\hat{\mathcal{F}}\Psi_{nm}(\theta, \omega) = \lambda_{nm}\Psi_{nm}(\theta, \omega)$. This operator is separable, and its eigenfunctions are of the form $\Psi_{nm}(\theta, \omega) = \exp(in\theta)\psi_{nm}(\omega)$. The functions $\psi_{nm}(\omega)$ satisfy

$$\gamma \frac{\partial}{\partial \omega}[\omega \psi_{nm}(\omega)] + D \frac{\partial^2}{\partial \omega^2} \psi_{nm}(\omega) - in\omega \psi_{nm}(\omega) = \lambda_{nm} \psi_{nm}(\omega). \quad (10)$$

To solve this equation, define the operators

$$\hat{F}_0 = \gamma \partial_x x + D \partial_x^2, \quad \hat{F}(\alpha) = \hat{F}_0 - \alpha x. \quad (11)$$

The operator $\hat{F}(\alpha)$ is not Hermitian. It is convenient to introduce the following transformation to a Hermitian form:

$$\hat{H}(\alpha) \equiv \exp(\gamma x^2/4D) \hat{F}(\alpha) \exp(-\gamma x^2/4D) = D \partial_x^2 - \frac{\gamma^2}{4D} x^2 + \frac{\gamma}{2} - \alpha x. \quad (12)$$

Note that $\hat{H}_0 \equiv \hat{H}(0)$ is an inverted harmonic oscillator, with eigenvalues $-\gamma m$, $m = 0, 1, \dots$. It will be convenient to use the Dirac notation for functions which are acted on by linear operators, replacing the usual angular ‘bra-kets’ with rounded ones to avoid confusion with our notation for expectation values. Thus the eigenfunctions of \hat{H}_0 are denoted by vectors $|\varphi_m\rangle$:

$$\hat{H}_0 |\varphi_m\rangle = -\gamma m |\varphi_m\rangle. \quad (13)$$

The eigenfunctions will be assumed to be normalised according to the quantum mechanical convention, so that the integral of their modulus squared over all space is equal to unity, for example the vector $|\varphi_0\rangle$ is a symbolic representation of the normalised eigenfunction $(\gamma/2\pi D)^{1/4} \exp(-\gamma x^2/4D)$. These eigenfunctions may be generated from $|\varphi_0\rangle$ by repeated application of creation and annihilation operators, \hat{a}^+ and \hat{a} respectively. These operators are

$$\hat{a}^+ = \sqrt{\frac{D}{\gamma}} \left(\frac{\gamma}{2D} x - \partial_x \right), \quad \hat{a} = \sqrt{\frac{D}{\gamma}} \left(\frac{\gamma}{2D} x + \partial_x \right). \quad (14)$$

Note that \hat{a}^+ is the Hermitian conjugate of \hat{a} . The annihilation and creation operators, \hat{a} , \hat{a}^+ satisfy

$$\hat{H}_0 = -\gamma \hat{a}^+ \hat{a}, \quad [\hat{H}_0, \hat{a}^+] = -\gamma \hat{a}^+, \quad [\hat{H}_0, \hat{a}] = \gamma \hat{a}, \quad [\hat{a}, \hat{a}^+] = 1. \quad (15)$$

These relations imply that if $\hat{H}_0 \varphi(x) = \lambda \varphi(x)$, then $\hat{a}^+ \varphi(x)$ and $\hat{a} \varphi(x)$ are also eigenfunctions, with eigenvalues $\lambda - \gamma$ and $\lambda + \gamma$ respectively (with the exception of the ground state, which is destroyed by \hat{a}). The following relations describe the action of \hat{a} , \hat{a}^+ on normalised eigenfunctions of \hat{H}_0 :

$$\hat{a} |\varphi_n\rangle = \sqrt{n} |\varphi_{n-1}\rangle, \quad \hat{a}^+ |\varphi_n\rangle = \sqrt{n+1} |\varphi_{n+1}\rangle. \quad (16)$$

The eigenfunctions of \hat{H}_0 are orthogonal, because this operator is Hermitian:

$$(\varphi_n | \varphi_m) \equiv \int_{-\infty}^{\infty} dx \varphi_n(x) \varphi_m(x) = \delta_{nm}. \quad (17)$$

The eigenfunctions and eigenvalues for other values of α can be obtained by considering a transformation to a new coordinate $y = x + x_0$, where the shift is $x_0 = 2D\alpha/\gamma^2$. In terms of this new coordinate,

we find that $\hat{H}(\alpha)$ is equivalent to $\hat{H}_0 + D\alpha^2/\gamma^2$. The eigenvalues $\lambda_m(\alpha)$ and eigenfunctions $|\varphi_m(\alpha)\rangle$ of $\hat{H}(\alpha)$ are therefore

$$\lambda_m(\alpha) = -\gamma m + \frac{D\alpha^2}{\gamma^2}, \quad |\varphi_m(\alpha)\rangle = \hat{T}(2D\alpha/\gamma^2) |\varphi_m\rangle \quad (18)$$

where $\hat{T}(X)$ is a translation operator, defined by $\hat{T}(X)f(x) = f(x - X)$. This operator may be represented as an exponential

$$\hat{T}(X) = \exp(-X\partial_x). \quad (19)$$

By expanding the exponential and comparing with the Taylor series, we see that this expression is consistent with the defining property that $\hat{T}(X)f(x) = f(x - X)$ for any function $f(x)$. Comparing (10) and (11), we see that we require $\alpha = in$, so that we shall require matrix elements of this operator for complex values of X .

The eigenfunctions $|\varphi_m\rangle$ of \hat{H}_0 correspond to eigenfunctions $\psi_m(x)$ of $\hat{F}(\alpha)$, where:

$$\psi_m(x) = \exp(-\gamma x^2/4D) \hat{T}(2D\alpha/\gamma^2) \varphi_m(x). \quad (20)$$

If a function $f(x)$ is written as a linear combination of these eigenfunctions

$$f(x) = \sum_{m=0}^{\infty} a_m \psi_m(x) \quad (21)$$

then, in order to use the orthogonality property (17), we must multiply $f(x)$ by $\exp(\gamma x^2/4D)$ and then act on it with $\hat{T}(-2D\alpha/\gamma^2)$, to obtain

$$a_m = \int_{-\infty}^{\infty} dx \varphi_m(x) \hat{T}(-2D\alpha/\gamma^2) \exp(\gamma x^2/4D) f(x). \quad (22)$$

In order to evaluate these coefficients, it will also be useful to have an expression for matrix elements of the translation operator (Franck-Condon factors) in the basis of the harmonic oscillator eigenfunctions:

$$I_{nm}(X) \equiv (\varphi_n | \hat{T}(X) | \varphi_m) \equiv \int_{-\infty}^{\infty} dx \varphi_n(x) \varphi_m(x - X). \quad (23)$$

By a simple adaptation of an argument presented in [5], for $n \geq m$ these can be shown to be given by

$$I_{nm}(X) = \sqrt{\frac{m!}{n!}} \left(\frac{X}{2} \sqrt{\frac{\gamma}{D}} \right)^{n-m} \exp\left(-\frac{X^2\gamma}{8D}\right) L_m^{(n-m)}\left(\frac{X^2\gamma}{4D}\right) \quad (24)$$

where $L_N^{(\alpha)}(x)$ is the associated Laguerre polynomial [6]:

$$L_N^{(\alpha)}(x) = \sum_{k=0}^N \frac{(-1)^k}{k!} \binom{n+\alpha}{n-k} x^k. \quad (25)$$

The case $n \leq m$ is obtained by using the fact that $I_{mn}(X) = I_{nm}(-X)$.

Using these results and noting that comparison of (10) and (11) implies $\alpha = in$, we see that the eigenfunctions of the Fokker-Planck operator $\hat{\mathcal{F}}$ are

$$\psi_{nm}(\theta, \omega) = \exp(in\theta) \exp(-\gamma\omega^2/4D) \hat{T}(2inD/\gamma^2) \varphi_m(\omega). \quad (26)$$

These eigenfunctions are complex-valued for $n \neq 0$, but note that the degenerate eigenfunctions $\psi_{n,m}$ and $\psi_{-n,m}$ are complex conjugates, so that we can form real-valued solutions of (9). A general solution of (9) can be written in the form

$$P(\theta, \omega, t) = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} a_{nm} \exp[-\gamma(m + n^2 D/\gamma^3)t] \psi_{nm}(\theta, \omega) \quad (27)$$

where the coefficients a_{nm} are determined by the initial conditions: using (22) gives

$$a_{nm} = \frac{1}{2\pi} \int_0^{2\pi} d\theta \exp(-in\theta) \int_{-\infty}^{\infty} d\omega \varphi_m(\omega) \hat{T}(-2inD/\gamma^2) \exp(\gamma\omega^2/4D) P(\theta, \omega, 0). \quad (28)$$

2.2 Evaluation of correlation functions

In order to evaluate the correlation function $C(t)$ defined by (6), we consider the distribution of θ with the initial condition that the initial orientation ($\theta = 0$, say) is known, but the angular momentum distribution initially in equilibrium:

$$P(\theta, \omega, 0) = \sqrt{\frac{\gamma}{2\pi D}} \exp(-\gamma\omega^2/2D) \delta(\theta) = \left(\frac{\gamma}{2\pi D}\right)^{1/4} \delta(\theta) \exp(-\gamma\omega^2/4D) \varphi_0(\omega) . \quad (29)$$

Using (28), the coefficients in (27) are

$$a_{nm} = \frac{1}{2\pi} \left(\frac{\gamma}{2\pi D}\right)^{1/4} \int_{-\infty}^{\infty} d\omega \varphi_m(\omega) \hat{T}(-2inD/\gamma^2) \varphi_0(\omega) = \frac{1}{2\pi} \left(\frac{\gamma}{2\pi D}\right)^{1/4} I_{m0}(-2inD/\gamma^2) . \quad (30)$$

The correlation function is

$$\begin{aligned} C(t) &= \langle \mathbf{n}(t) \cdot \mathbf{n}(0) \rangle = \langle \cos \theta \rangle \\ &= \int_0^{2\pi} d\theta \cos \theta \int_{-\infty}^{\infty} d\omega P(\theta, \omega, t) \\ &= 2\pi \exp(-Dt/\gamma^2) \sum_{m=0}^{\infty} \text{Re} \left[a_{1m} \exp(-\gamma mt) \int_{-\infty}^{\infty} d\omega \exp(-\gamma\omega^2/4D) \hat{T}(2iD/\gamma^2) \varphi_m(\omega) \right] \\ &= \exp(-Dt/\gamma^2) \sum_{m=0}^{\infty} I_{m0}(-2iD/\gamma^2) I_{0m}(2iD/\gamma^2) \exp(-\gamma mt) . \end{aligned} \quad (31)$$

Note that (24) implies that the only associated Laguerre polynomials which are required are $L_0^{(\alpha)}(x) = 1$, so that the Franck-Condon factors in (31) are

$$I_{m0}(2iD/\gamma^2) = \sqrt{\frac{1}{m!}} (-i\beta)^m \exp(\beta^2/2) = I_{0m}(-2iD/\gamma^2) \quad (32)$$

where $\beta = \sqrt{D/\gamma^3}$ is the dimensionless parameter defined in (5), so that the correlation function in (31) is

$$C(t) = \exp(-Dt/\gamma^2) \exp(\beta^2) \sum_{m=0}^{\infty} \frac{(-\beta^2)^m}{m!} \exp(-m\gamma t) . \quad (33)$$

Using the identity

$$\sum_{m=0}^{\infty} \frac{(-x)^m}{m!} \exp(-am) = \exp[-x \exp(-a)] \quad (34)$$

the correlation function can be obtained in closed form:

$$C(t) = c(\gamma t, \sqrt{D/\gamma^3}) , \quad c(\tau, \beta) = \exp[\beta^2(1 - \tau - \exp(-\tau))] . \quad (35)$$

Figure 1 shows a comparison between the correlation function obtained by numerical averaging for the circular Ornstein-Uhlenbeck process, for three different values of β . In each case the results are compared with the theoretical expression, equation (35), and the agreement is excellent.

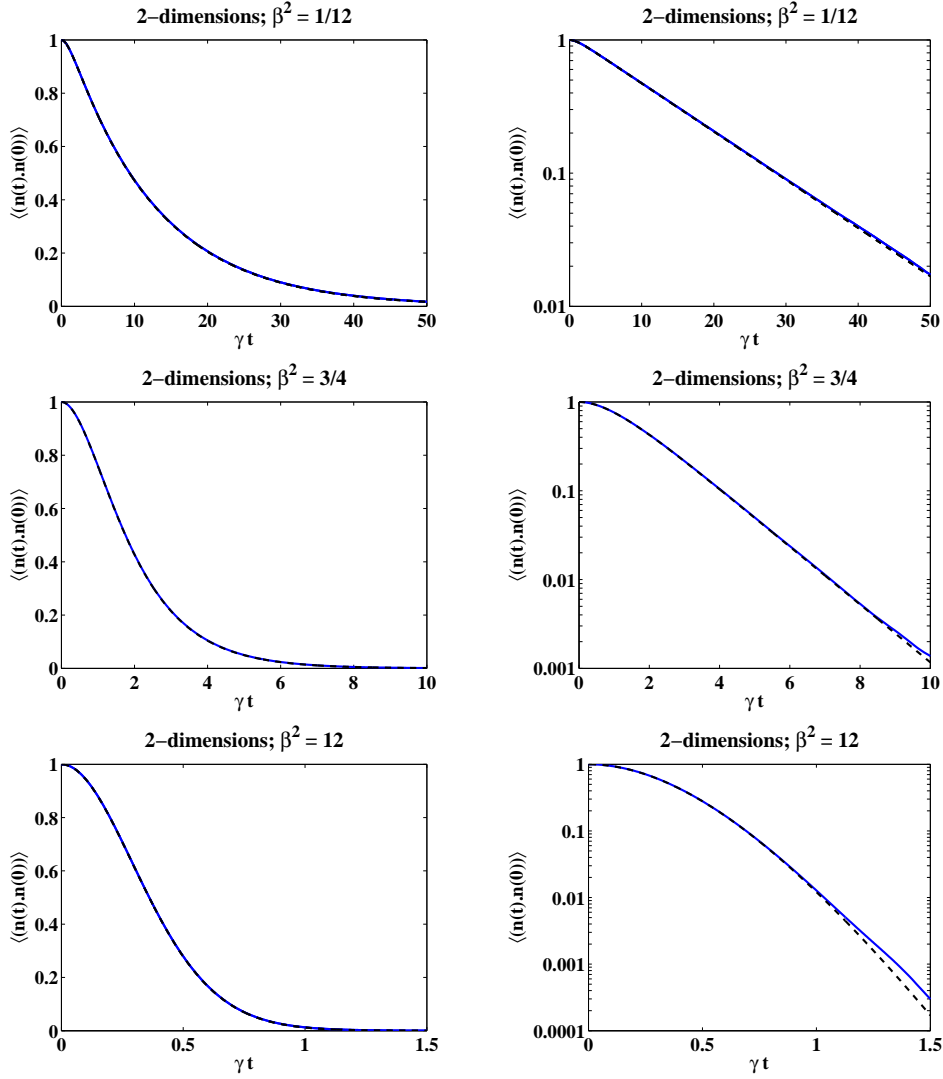


Fig. 1 Correlation function for the circular Ornstein-Uhlenbeck process, for three values of $\beta^2 = D/\gamma^3$: $\beta^2 = 1/12$ (upper row), $\beta^2 = 3/4$ (middle row) and $\beta^2 = 12$ (lower row). The simulations (full curves) show excellent agreement with the theoretical results, equation (35), shown as a dashed line.

2.3 Discussion

In the limits $\beta \rightarrow 0$ and $\beta \rightarrow \infty$ the correlation function approaches limiting forms which are exponential and Gaussian, respectively: from (35) we find

$$C(t) \sim \exp(-Dt/\gamma^2), \quad \beta \ll 1 \quad (36)$$

$$C(t) \sim \exp(-Dt^2/2\gamma), \quad \beta \gg 1. \quad (37)$$

It is instructive to consider how these limiting cases arise. In the case where $\sqrt{D/\gamma^3} \ll 1$, the partial probability density $P(\theta, t)$ satisfies a diffusion equation

$$\frac{\partial P}{\partial t} = D \frac{\partial^2 P}{\partial \theta^2} \quad (38)$$

which is to be solved with the initial condition $P(\theta, 0) = \delta(\theta)$. The solution of this equation on the circle is

$$P(\theta, t) = \sum_{m=-\infty}^{\infty} a_m \exp(im\theta) \exp(-m^2 \mathcal{D}t) . \quad (39)$$

The initial condition gives $a_m = 1/2\pi$, so that

$$\begin{aligned} \langle \cos \theta(t) \rangle &= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \exp(-m^2 \mathcal{D}t) \int_0^{2\pi} d\theta \cos(\theta) \exp(im\theta) \\ &= \exp(-\mathcal{D}t) = \exp(-Dt/\gamma^2) \end{aligned} \quad (40)$$

in agreement with (36). In the opposite limit, $\sqrt{D/\gamma^3} \gg 1$, the particle rotates around the circle at a rate which is equal to its initial angular velocity. The equilibrium distribution of angular momentum has density

$$P(\omega) = \sqrt{\frac{\gamma}{2\pi D}} \exp(-\omega^2 \gamma / 2D) . \quad (41)$$

After time t the angle is $\theta = \omega t$, so that the probability distribution of the angle is

$$P(\theta, t) = \sqrt{\frac{\gamma}{2\pi D t^2}} \exp(-\theta^2 \gamma / 2D t^2) . \quad (42)$$

The correlation function is then

$$\begin{aligned} \langle \cos \theta(t) \rangle &= \sqrt{\frac{\gamma}{2\pi D t^2}} \int_{-\infty}^{\infty} d\theta \cos(\theta) \exp(-\theta^2 \gamma / 2D t^2) \\ &= \exp(-Dt^2 / 2\gamma) \end{aligned} \quad (43)$$

in agreement with (37).

The introduction mentioned that there is a surprising aspect to the behaviour of the series (33) in the limit as $\beta \rightarrow \infty$. Despite the fact that every term in this series has an exponential decay with a timescale shorter than $1/\gamma\beta^2$, the exact evaluation of the sum of this series, equation (35), decays on a much longer timescale, $1/(\gamma\beta)$.

3 Three dimensional case

3.1 Formulation and Fokker-Planck equation

We consider a unit vector $\mathbf{n}(t)$ evolving according to the equation:

$$\frac{d\mathbf{n}}{dt} = \boldsymbol{\omega} \wedge \mathbf{n} \quad (44)$$

where $\boldsymbol{\omega}$ is the angular velocity vector. The components ω_i of the angular velocity are determined by independent Ornstein-Uhlenbeck equations:

$$\frac{d\omega_i}{dt} = -\gamma\omega_i + \sqrt{2D}\eta_i(t) \quad (45)$$

where the $\eta_i(t)$ are independent white noise signals, with statistics specified by equation (2). This process is described by a Fokker-Planck equation for the joint probability density of \mathbf{n} and $\boldsymbol{\omega}$. The general form for the Fokker-Planck equation in a space with coordinates x_i is [3]:

$$\frac{\partial P}{\partial t} = - \sum_{i=1}^6 \frac{\partial}{\partial x_i} (v_i P) + \sum_{i=1}^6 \sum_{j=1}^6 \frac{\partial^2}{\partial x_i \partial x_j} (D_{ij} P) . \quad (46)$$

The velocities v_i and diffusion coefficients D_{ij} are defined in terms of the expectation values of increments δx_i in time δt by writing $\langle \delta x_i \rangle = v_i \delta t$ and $\langle \delta x_i \delta x_j \rangle = 2D_{ij} \delta t$. In our case $(x_1, x_2, \dots, x_6) = \mathbf{x} = (\mathbf{n}, \boldsymbol{\omega}) = (n_1, n_2, n_3, \omega_1, \omega_2, \omega_3)$. The velocities are

$$v_i = \epsilon_{ijk} \omega_j n_k, \quad i = 1, 2, 3, \quad v_i = -\gamma \omega_i, \quad i = 4, 5, 6 \quad (47)$$

and the diffusion coefficients are

$$D_{ij} = D \delta_{ij}, \quad i = 4, 5, 6, \quad D_{ij} = 0, \quad i = 1, 2, 3. \quad (48)$$

The Fokker-Planck equation is therefore

$$\frac{\partial P}{\partial t} = \hat{\mathcal{F}}P = \epsilon_{ijk} \frac{\partial}{\partial n_i} (n_j \omega_k P) + \gamma \frac{\partial}{\partial \omega_i} (\omega_i P) + D \frac{\partial^2 P}{\partial \omega_i \partial \omega_i} \quad (49)$$

where in equation (49), as well as in the equations below, repeated indices are summed over the values 1, 2, 3. The Fokker-Planck operator can also be expressed in the form

$$\hat{\mathcal{F}} = \gamma \partial_i \omega_i + D \partial_i \partial_i - \omega_i \hat{J}_i \quad (50)$$

where \hat{J}_i are components of an angular momentum operator, defined by

$$\hat{J}_i = \epsilon_{ijk} n_j \frac{\partial}{\partial n_k} \quad (51)$$

and where $\partial_i = \partial / \partial \omega_i$. Note that this definition differs from that which is commonly used in quantum mechanics texts (such as [7]) by a factor of $i = \sqrt{-1}$. The Fokker-Planck operator (50) can also be expressed in the form:

$$\hat{\mathcal{F}} = \hat{\mathcal{F}}_0 - \boldsymbol{\omega} \cdot \hat{\mathbf{J}}, \quad \hat{\mathcal{F}}_0 = \gamma \boldsymbol{\nabla} \cdot \boldsymbol{\omega} + D \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}. \quad (52)$$

where $\boldsymbol{\nabla}$ is the gradient in the angular momentum space.

The variables may be made dimensionless by using:

$$\bar{t} \equiv \gamma t \quad (53)$$

$$\bar{\omega}_i \equiv \sqrt{\frac{\gamma}{2D}} \omega_i. \quad (54)$$

In dimensionless form the Fokker-Planck equation reads:

$$\partial_{\bar{t}} P = \partial_i (\omega_i P) + \sqrt{2} \beta \epsilon_{ijk} \omega_k \partial_{n_i} (n_j P) + \frac{1}{2} \partial_i^2 P \quad (55)$$

where the variables in equation (55) are dimensionless, as defined in equations (53,54); for simplicity, the overbars have been omitted.

3.2 Symmetry analysis

Consider the symmetry properties of the Ornstein-Uhlenbeck operator describing the evolution of the joint probability density function $P(\mathbf{n}, \boldsymbol{\omega}, t)$. The physical properties of the system are invariant under the rotation group, in the sense that an arbitrary rotation of both \mathbf{n} and $\boldsymbol{\omega}$ leaves the problem unchanged. We define angular momentum operators which generate these rotations:

$$\begin{aligned} \hat{L}_i &= \epsilon_{ijk} \omega_j \partial_{\omega_k} \\ \hat{J}_i &= \epsilon_{ijk} n_j \partial_{n_k} \end{aligned} \quad (56)$$

(the operators \hat{J}_i were already considered in (51)). These operators are related to the operators used in quantum mechanics by a factor i , that is $\hat{\mathbf{L}} = i \hat{\mathbf{L}}_{\text{QM}}$ where $\hat{\mathbf{L}}_{\text{QM}}$ is the angular momentum operator defined in standard texts such as [7]. Thus, adapting standard results [7], the commutation relations of the operators are:

$$[\hat{L}_i, \hat{L}_j] = -\epsilon_{ijk} \hat{L}_k \quad (57)$$

and similar relations for \hat{J}_i . It is also useful to note that:

$$[\hat{L}_i, \omega_j] = -\epsilon_{ijk} \omega_k . \quad (58)$$

The term $\boldsymbol{\omega} \cdot \hat{\mathbf{J}}$ is invariant when rotating *simultaneously* \mathbf{n} and $\boldsymbol{\omega}$, and using (57) and (58) it is easy to check that:

$$[\hat{\mathbf{J}} + \hat{\mathbf{L}}, \boldsymbol{\omega} \cdot \hat{\mathbf{J}}] = 0 . \quad (59)$$

In the same way, it is easy to see that

$$[\hat{\mathbf{J}}^2, \boldsymbol{\omega} \cdot \hat{\mathbf{J}}] = 0 . \quad (60)$$

It is clear that these are also symmetries of the other elements of the Fokker-Planck operator in (55). Because the system is invariant under rotation simultaneously of $\boldsymbol{\omega}$ and \mathbf{n} , the Fokker-Planck evolution operator commutes with the *total* angular momentum, as well as with the magnitude of the angular momentum of $\mathbf{n}(t)$. The set of conserved quantities is therefore

$$\hat{\mathbf{K}} = \hat{\mathbf{L}} + \hat{\mathbf{J}} , \quad \hat{\mathbf{J}}^2 \quad (61)$$

where $\hat{\mathbf{L}}$ acts on the $\boldsymbol{\omega}$ variables and $\hat{\mathbf{J}}$ acts on the \mathbf{n} variables. The corresponding quantum numbers are denoted j, m_j for $\hat{\mathbf{J}}$, l, m_l for $\hat{\mathbf{L}}$, and k, m_k for $\hat{\mathbf{K}}$.

Thus, the Fokker-Planck operator has a block-diagonal structure, where the blocks are labelled by a given set of values of j, k and m_k , where $-j(j+1)$ is the eigenvalue of $\hat{\mathbf{J}}^2$, m_k the eigenvalue of $\hat{\mathbf{K}}_z$ and $-k(k+1)$ the eigenvalue of $\hat{\mathbf{K}}^2$. These symmetries indicate that spherical polar coordinates and spherical harmonic functions will prove useful. We use (θ, ϕ) as the polar angles for \mathbf{n} and (ω, θ', ϕ') as the spherical polar coordinates of $\boldsymbol{\omega}$. The spherical harmonic functions $Y_{lm}(\theta, \phi)$ will be denoted by Dirac state vectors, writing $|Y_{lm})$ when the arguments are (θ, ϕ) and $|Y'_{lm})$ when the arguments are (θ', ϕ') .

Further constraints due to symmetry considerations can follow from the initial conditions. For example, the initial condition for evaluation of correlation functions has the variable $\boldsymbol{\omega}$ in the equilibrium state, which is spherically symmetric with $l = 0$ (and which is easily seen to be a Gaussian function of $\boldsymbol{\omega}$). The value of \mathbf{n} is set equal to one particular vector, say \mathbf{e}_z . The initial condition is therefore

$$P(\mathbf{n}, \boldsymbol{\omega}, 0) = \sqrt{4\pi} \left(\frac{\gamma}{2\pi D} \right)^{3/2} \exp(-\gamma \omega^2 / 2D) Y_{00}(\theta', \phi') \delta(\mathbf{n} - \mathbf{e}_z) . \quad (62)$$

The δ -function distribution can be resolved into a sum the spherical harmonics, with $j = 0, 1, 2, \dots$ but with $m_j = 0$ in each case:

$$\delta(\mathbf{n} - \mathbf{e}_z) = \sum_{j=0}^{\infty} \frac{2j+1}{4\pi} P_j(\mathbf{n} \cdot \mathbf{e}_z) \quad (63)$$

where the functions $P_j(x)$ are Legendre polynomials. This means that we can confine attention to the $m_k = 0$ subspace when evaluating an equilibrium correlation function of $\mathbf{n}(t)$.

Furthermore, in the case of the simplest correlation function, $C(t) = \langle \cos \theta \rangle$, the quantity being averaged spans only the $j = 1$ subspace, so we can confine our attention to the $j = 1$ subspace. Thus in order to evaluate $C(t)$ we must consider the $j = 1, m_k = 0$ subspace. Furthermore, the initial condition has $l = 0$, so the total angular momentum is $k = 1$. The values of \mathbf{K}^2 and \mathbf{J}^2 are constants of the motion, indicating that we must consider only solutions with $k = 1$ and $j = 1$. Setting $k = 1$ and $j = 1$, the triangle relation for $\mathbf{L} = \mathbf{K} - \mathbf{J}$ is $|k - j| \leq l \leq k + j$, that is, $l = 0, 1, 2$.

We can construct functions of the angular variables with definite values of \mathbf{J}^2 , \mathbf{L}^2 , \mathbf{K}^2 and K_z , labelled by quantum numbers j, l, k, m_k . Let these functions be denoted by $\mathcal{Y}_{j,l,k,m_k}(\theta, \phi, \theta', \phi')$, and we assume that these functions are normalised so that they form an orthonormal set, with the usual integration measure for a cartesian product of two spherical surfaces. The symmetry considerations discussed above imply that the solution in the $j = 1$ subspace may be written in the form

$$P_1(\mathbf{n}, \boldsymbol{\omega}, t) = \sum_{l=0}^2 \psi_l(\omega, t) \mathcal{Y}_{1,l,1,0}(\theta, \phi, \theta', \phi') . \quad (64)$$

This result shows that, if we are concerned with evaluating the correlation function $C(t) = \langle \cos \theta \rangle$, then symmetry considerations reduce the Fokker-Planck equation to a system of three coupled ordinary differential equations. More generally, the calculation of correlation functions such as $\langle P_j(\cos \theta) \rangle$ implies values of l in the range $0 \leq l \leq 2j$, so the solution requires $2j + 1$ functions of ω .

3.3 Harmonic oscillator basis

It will also be useful to consider the exact solution of the Ornstein-Uhlenbeck process describing the evolution of the angular momentum $\boldsymbol{\omega}$, independent of evolution of \mathbf{n} . This will be related to the three-dimensional quantum spherical harmonic oscillator.

The operator $\hat{\mathcal{F}}_0$ in (52) has a structure which is closely related to the harmonic oscillator of quantum mechanics. It is convenient to transform the Fokker-Planck operator $\hat{\mathcal{F}}_0$ into a three-dimensional isotropic harmonic oscillator. We consider the operator

$$\hat{\mathcal{H}} = \exp(\chi/2) \hat{\mathcal{F}} \exp(-\chi/2), \quad \chi = \frac{\gamma(\omega_1^2 + \omega_2^2 + \omega_3^2)}{2D}. \quad (65)$$

Using the dimensionless variables defined by (53) and (54), we can express $\hat{\mathcal{H}}$ in the form

$$\hat{\mathcal{H}} = -\gamma \left[\hat{\mathbf{a}}^+ \cdot \hat{\mathbf{a}} + \sqrt{2}\beta \hat{\mathbf{J}} \cdot \boldsymbol{\omega} \right] \equiv \hat{\mathcal{H}}_0 - \sqrt{2}\beta\gamma \hat{\mathbf{J}} \cdot \boldsymbol{\omega} \quad (66)$$

where the components of $\hat{\mathbf{a}}^+$ and $\hat{\mathbf{a}}$ are, respectively, creation and annihilation operators for the i degree of freedom, using the dimensionless variables defined by equations (53, 54) :

$$\hat{a}_i^+ = \frac{1}{\sqrt{2}} (\omega_i - \partial_i), \quad \hat{a}_i = \frac{1}{\sqrt{2}} (\omega_i + \partial_i). \quad (67)$$

The eigenvalues of $\hat{\mathcal{H}}_0$ are $-\gamma(k_1 + k_2 + k_3)$, where $k_i = 0, 1, 2, \dots$. Note that all of the eigenvalues except the ground state are degenerate.

As well as being separable in Cartesian coordinates, $\hat{\mathcal{H}}_0$ is also separable in spherical polar coordinates:

$$\hat{\mathcal{H}}_0 = \gamma \left[\frac{1}{2} \nabla_{\boldsymbol{\omega}}^2 - \frac{1}{2} \omega^2 + \frac{3}{2} \right] \quad (68)$$

where the Laplacian operator may be expressed as

$$\nabla_{\boldsymbol{\omega}}^2 \equiv \partial_{\omega}^2 + \frac{2}{\omega} \partial_{\omega} + \frac{1}{\omega^2} \hat{\mathbf{L}}^2 \quad (69)$$

and $\hat{\mathbf{L}}$ is, up to a factor i , the usual angular momentum operator acting on \mathbf{e}_{ω} ; its eigenvalues are $-l(l+1)$, $l = 0, 1, 2, \dots$. The degenerate multiplets can, therefore, also be resolved as states which are eigenfunctions of $\hat{\mathbf{L}}^2$ and \hat{L}_z , labelled by quantum numbers n, l, m (where $-l(l+1)$ and m are eigenvalues of $\hat{\mathbf{L}}^2$ and \hat{L}_z respectively). The ground state eigenfunction of $\hat{\mathcal{H}}_0 = -\gamma \hat{\mathbf{a}}^+ \cdot \hat{\mathbf{a}}$ is

$$\varphi_{000}(\boldsymbol{\omega}) = \left(\frac{1}{2\pi} \right)^{3/4} \exp \left(-\frac{\omega^2}{2} \right) \quad (70)$$

and the other eigenfunctions of $\hat{\mathcal{H}}_0$ are of the form

$$\varphi_{nlm}(\boldsymbol{\omega}) = \sqrt{4\pi} Y_{lm}(\theta', \phi') \omega^l P_{nl}(\omega^2) \varphi_{000}(\boldsymbol{\omega}) \quad (71)$$

where $P_{nl}(x)$ is a polynomial of degree n , proportional to the generalized Laguerre polynomial $L_n^{(l+1/2)}$ [7]. The eigenfunctions are normalised in the usual way:

$$\begin{aligned} \delta_{nn'} &= \int_0^\infty dx \, x^{2+2l} \exp(-x^2) P_{nl}(x^2) P_{n'l}(x^2) \\ &= \frac{1}{2} \int_0^\infty dy \, y^{l+1/2} \exp(-y) P_{nl}(y) P_{n'l}(y). \end{aligned} \quad (72)$$

Using the orthonormality relation for the generalized Laguerre polynomials:

$$\int_0^\infty dx x^\alpha \exp(-x) L_n^{(\alpha)}(x) L_m^{(\alpha)}(x) = \delta_{nm} \frac{\Gamma(n + \alpha + 1)}{n!} . \quad (73)$$

we deduce the relation between P_{nl} and the generalized Laguerre polynomials:

$$P_{nl}(x) = \mathcal{N}_{nl} L_n^{(l+\frac{1}{2})}(x) , \quad \mathcal{N}_{nl} = \sqrt{\frac{2n!}{(n+l+\frac{1}{2})\Gamma(n+l+\frac{1}{2})}} . \quad (74)$$

3.4 Equations of motion for $j = 1$ modes

Here we consider how to write an equation of motion for the projection onto the $j = 1$ modes, which contribute to the correlation function $C(t)$. The harmonic oscillator basis which was introduced in section 3.3 will prove useful here.

In section 3.2 we showed how symmetry considerations constrain the angular dependences of the solutions. The angle-dependent parts of the solution are constructed from functions with known values of \mathbf{J}^2 , \mathbf{L}^2 , \mathbf{K}^2 and K_z , with quantum numbers j, l, k, m_k . These functions may be expressed in terms of tensor products of spherical harmonics, writing

$$|\mathcal{Y}_{j,l,k,m_k}\rangle = \sum_{l_1,m_1} \sum_{l_2,m_2} (l_1, m_1; l_2, m_2 | j, l, k, m_k) |Y_{l_1,m_1}\rangle \otimes |Y'_{l_2,m_2}\rangle \quad (75)$$

where $|Y_{l_1,m_1}\rangle$ represents the spherical harmonic $Y_{l_1,m_1}(\theta, \phi)$ which is a function of polar angles representing the direction of \mathbf{n} , and $|Y'_{l_2,m_2}\rangle$ represents $Y_{l_2,m_2}(\theta', \phi')$, which is a function of the polar angles for $\boldsymbol{\omega}$. The coefficients $(l_1, m_1; l_2, m_2 | j, l, k, m_k)$ are termed Clebsch-Gordon coefficients (see, for example, [8, 7]), and this representation is useful because the spherical harmonics have well-known and convenient properties.

In particular, determining the correlation function $C(t)$ requires a solution involving just three functions $\psi_l(\omega, t)$, multiplying the angular functions $\mathcal{Y}_{l,l,1,0}$, with $l = 0, 1, 2$ (see equation (64)). From tabulations of Clebsch-Gordon coefficients we find:

$$\begin{aligned} |\mathcal{Y}_{1,0,1,0}\rangle &= |Y_{10}\rangle \otimes |Y'_{00}\rangle \\ |\mathcal{Y}_{1,1,1,0}\rangle &= \sqrt{\frac{1}{2}} |Y_{1,+1}\rangle \otimes |Y'_{1,-1}\rangle - \sqrt{\frac{1}{2}} |Y_{1,-1}\rangle \otimes |Y'_{1,+1}\rangle \\ |\mathcal{Y}_{1,2,1,0}\rangle &= \sqrt{\frac{3}{10}} |Y_{2,1}\rangle \otimes |Y'_{1,-1}\rangle - \sqrt{\frac{2}{5}} |Y_{2,0}\rangle \otimes |Y'_{1,0}\rangle + \sqrt{\frac{3}{10}} |Y_{2,-1}\rangle \otimes |Y'_{1,+1}\rangle . \end{aligned} \quad (76)$$

In order to express the equation of motion in the $j = 1$ subspace, it is necessary to rewrite the operator $\boldsymbol{\omega} \cdot \mathbf{J}$ so that its action upon the spherical harmonics is explicit. To this end, consider the angular momentum ladder operators:

$$\hat{J}_+ = (\hat{J}_x + i\hat{J}_y) , \quad \hat{J}_- = (\hat{J}_x - i\hat{J}_y) \quad (77)$$

It is straightforward to see that:

$$[\hat{J}_z, \hat{J}_\pm] = \pm i\hat{J}_\pm . \quad (78)$$

Thus, applying the operator \hat{J}_+ to the eigenstate of \hat{J}_z with a quantum number m , namely the spherical harmonic $|Y_{l,m}\rangle$, one finds $\hat{J}_z \hat{J}_\pm |Y_{l,m}\rangle = i(m \pm 1) \hat{J}_\pm |Y_{l,m}\rangle$. The operator \hat{J}_+ thus increases the azimuthal quantum number by one, whereas \hat{J}_- decreases the azimuthal quantum number by one. For completeness, the prefactor, up to a phase, can be obtained by expressing $\hat{\mathbf{J}}^2$ as:

$$\hat{\mathbf{J}}^2 = \hat{J}_+ \hat{J}_- + \hat{J}_z^2 - i\hat{J}_z = \hat{J}_- \hat{J}_+ + \hat{J}_z^2 + i\hat{J}_z \quad (79)$$

which immediately leads to:

$$\begin{aligned}\hat{J}_+|Y_{l,m-1}\rangle &= i\sqrt{(l+m)(l-m+1)}|Y_{l,m}\rangle \\ \hat{J}_-|Y_{l,m}\rangle &= i\sqrt{(l+m)(l-m+1)}|Y_{l,m-1}\rangle.\end{aligned}\quad (80)$$

With these results in place, we can express $\boldsymbol{\omega} \cdot \hat{\mathbf{J}}$ in terms of the operators \hat{J}_\pm . Elementary algebra leads to:

$$\boldsymbol{\omega} \cdot \hat{\mathbf{J}} = \frac{1}{2}(\omega_x - i\omega_y)\hat{J}_+ + \frac{1}{2}(\omega_x + i\omega_y)\hat{J}_- + \omega_z\hat{J}_z. \quad (81)$$

If one notices further that:

$$\begin{aligned}(\omega_x + i\omega_y) &= -\omega\sqrt{\frac{8\pi}{3}}Y_{1,1}(\theta', \phi') \\ (\omega_x - i\omega_y) &= +\omega\sqrt{\frac{8\pi}{3}}Y_{1,-1}(\theta', \phi') \\ \omega_z &= w\sqrt{\frac{4\pi}{3}}Y_{1,0}(\theta', \phi')\end{aligned}\quad (82)$$

then, equation (81) leads to:

$$\boldsymbol{\omega} \cdot \mathbf{J} = \omega\sqrt{\frac{4\pi}{3}}\left(\frac{1}{\sqrt{2}}Y_{1,-1}(\theta', \phi')\hat{J}_+ + Y_{1,0}(\theta', \phi')\hat{J}_z - \frac{1}{\sqrt{2}}Y_{1,1}(\theta', \phi')\hat{J}_-\right). \quad (83)$$

The formulation of equation (83) is useful to understand how the operator $\boldsymbol{\omega} \cdot \hat{\mathbf{J}}$ couples modes to each other. In particular, with the help of equations (83) and (76) we can determine the matrix elements

$$\langle \mathcal{T}_{1,i,1,0} | \boldsymbol{\omega} \cdot \hat{\mathbf{J}} | \mathcal{T}_{1,j,1,0} \rangle \equiv \omega \mathcal{A}_{ij} \quad (84)$$

where $i, j \in \{0, 1, 2\}$. Only four of these matrix elements are non-zero. After a lengthy but mechanical calculation we find the following values for the non-zero matrix elements:

$$\begin{aligned}\mathcal{A}_{01} &= \sqrt{\frac{2}{3}} & \mathcal{A}_{10} &= -\sqrt{\frac{2}{3}} \\ \mathcal{A}_{12} &= \sqrt{\frac{1}{3}} & \mathcal{A}_{21} &= -\sqrt{\frac{1}{3}}.\end{aligned}\quad (85)$$

It is convenient to use the connection with the spherical harmonic oscillator, and to replace the functions $\psi_l(\omega, t)$ in (64) by

$$\zeta_l(\omega, t) = \exp(-\omega^2/4) \psi_l(\omega, t) \quad (86)$$

(here we use the dimensionless variables (53), (54)). Substituting (64), (86) into the Fokker-Planck equation, multiplying by $|\mathcal{T}_{1,j,1,0}\rangle$, and integrating over the product of two spheres, we obtain three partial differential equations for the three components coupling to the $j = 1$ mode. These equations are:

$$\begin{aligned}\frac{\partial}{\partial t}\zeta_0(\omega, t) &= \hat{L}_0\zeta_0(\omega, t) + \sqrt{2}\beta\omega\mathcal{A}_{01}\zeta_1(\omega, t) \\ \frac{\partial}{\partial t}\zeta_1(\omega, t) &= \hat{L}_1\zeta_1(\omega, t) + \sqrt{2}\beta\omega[\mathcal{A}_{10}\zeta_0(\omega, t) + \mathcal{A}_{12}\zeta_2(\omega, t)] \\ \frac{\partial}{\partial t}\zeta_2(\omega, t) &= \hat{L}_2\zeta_2(\omega, t) + \sqrt{2}\beta\omega\mathcal{A}_{21}\zeta_1(\omega, t)\end{aligned}\quad (87)$$

where

$$\hat{L}_j \equiv \frac{1}{2}\left[\partial_\omega^2 + \frac{2}{\omega}\partial_\omega - \frac{j(j+1)}{\omega^2} - \omega^2 + 3\right]. \quad (88)$$

4 Asymptotic properties of the correlation function

In three dimensions we are only able to determine the spectrum of $\hat{\mathcal{H}}$ (defined by equation (66)) by analytical methods in the limits $\beta \rightarrow 0$ and $\beta \rightarrow \infty$. This section considers various asymptotic approximations for the correlation function.

4.1 Short-time limit

The correlation function $C(t) = \langle \mathbf{n}(t) \cdot \mathbf{n}(0) \rangle$ can be calculated by using the initial condition $\mathbf{n}(0) = \mathbf{e}_3$ and then computing $\langle \cos \theta \rangle$. For short times, the polar angle is approximated by $\theta = (\omega_1^2 + \omega_2^2)^{1/2} t + O(t^2)$. In the short-time limit, therefore

$$C(t) = \langle 1 - \frac{1}{2}\theta^2 + \dots \rangle = 1 - \frac{1}{2}\langle \omega_1^2 + \omega_2^2 \rangle t^2 + \dots \quad (89)$$

Using equation (3), we have $\langle \omega_i^2 \rangle = D/\gamma$. The leading order behaviour of the correlation function is therefore

$$C(t) = 1 - \frac{Dt^2}{\gamma} + O(t^3) = 1 - \beta^2 \bar{t}^2 + O(t^3) \quad (90)$$

where \bar{t} is the dimensionless time defined by (53). This result is valid for all β .

4.2 Diffusive ($\beta \rightarrow 0$) limit

In the limit $D/\gamma^3 \ll 1$, the angle of \mathbf{n} diffuses with diffusion coefficient $\mathcal{D} = D/\gamma^2$. The solution of the diffusion equation on the surface of a sphere is expressed in terms of spherical harmonics $Y_{lm}(\theta, \phi)$, for which the eigenvalues of the Laplacian are $-l(l+1)$:

$$P(\theta, \phi, t) = \sum_{l=0}^{\infty} \sum_{m=-l}^l a_{lm} \exp[-l(l+1)Dt/\gamma^2] Y_{lm}(\theta, \phi) . \quad (91)$$

The required coefficients are obtained using orthogonality of spherical harmonics: $a_{lm} = Y_{l0}(\mathbf{e}_3)\delta_{m0}$, so that $a_{lm} = \delta_{m0}\delta_{l1}\sqrt{3/4\pi}$. The expectation value of $\cos(\theta)$ therefore has a simple exponential decay:

$$\langle \cos \theta \rangle = \sqrt{\frac{4\pi}{3}} \langle Y_{10}(t) \rangle = \exp(-2Dt/\gamma^2) = \exp(-2\beta^2 \bar{t}) . \quad (92)$$

This approximation does not have the correct limiting behaviour as $t \rightarrow 0$, which is given by (90). The following approximation to the numerically determined correlation function approaches (92) at $\beta \rightarrow 0$ for all t and has the correct quadratic behaviour at $t = 0$:

$$C(t) \approx \exp\left(-\frac{2\beta^2 \bar{t}^2}{\sqrt{4 + \bar{t}^2}}\right) . \quad (93)$$

4.3 Large β limit

In the limit $D/\gamma^3 \gg 1$, the short-time behaviour of the correlation function is determined by tumbling motion with fixed angular momentum. Consider the solution of the equation of motion $\dot{\mathbf{n}} = \boldsymbol{\omega} \wedge \mathbf{n}$ in the case where $\boldsymbol{\omega} = \omega \mathbf{e}_\omega$ is constant and where the initial direction is $\mathbf{n}_0 = \mathbf{e}_3$. The solution is

$$\mathbf{n}(t) = \mathbf{a} + \mathbf{b} \cos \omega t + \mathbf{c} \sin \omega t \quad (94)$$

where

$$\mathbf{a} = (\mathbf{e}_\omega \cdot \mathbf{n}_0) \mathbf{e}_\omega , \quad \mathbf{b} = \mathbf{n}_0 - \mathbf{a} , \quad \mathbf{c} = \frac{\mathbf{a} \wedge \mathbf{b}}{|\mathbf{a} \wedge \mathbf{b}|} |\mathbf{b}| \quad (95)$$

are three mutually orthogonal vectors. It follows that

$$\mathbf{n}(t) \cdot \mathbf{n}(0) = |\mathbf{n}_0 \cdot \mathbf{e}_\omega|^2 + \cos \omega t [1 - |\mathbf{n}_0 \cdot \mathbf{e}_\omega|^2] . \quad (96)$$

Now integrate over the distribution of angular momentum to obtain:

$$\begin{aligned} \langle \mathbf{n}(t) \cdot \mathbf{n}(0) \rangle &= \int d\boldsymbol{\omega} P(\boldsymbol{\omega}) \left[\frac{\omega_z^2}{\omega^2} + \cos \omega t \left(1 - \frac{\omega_z^2}{\omega^2} \right) \right] \\ &= \left(\frac{\gamma}{2\pi D} \right)^{3/2} \int_{-\infty}^{\infty} d\omega_x \int_{-\infty}^{\infty} d\omega_y \int_{-\infty}^{\infty} d\omega_z \exp(-\gamma \boldsymbol{\omega}^2 / 2D) \left[\frac{\omega_z^2}{\omega^2} + \left(1 - \frac{\omega_z^2}{\omega^2} \right) \cos \omega t \right] \\ &= \left(\frac{\gamma}{2\pi D} \right)^{3/2} \int_0^{\infty} d\omega \int_0^{\pi} d\theta \, 2\pi \omega^2 \sin \theta \exp(-\gamma \omega^2 / 2D) [\cos^2 \theta + \sin^2 \theta \cos \omega t] \\ &= 4\pi \left(\frac{\gamma}{2\pi D} \right)^{3/2} \int_0^{\infty} d\omega \, \omega^2 \exp(-\gamma \omega^2 / 2D) \left[\frac{1}{3} + \frac{2}{3} \cos \omega t \right] \\ &= \frac{1}{3} + \frac{2}{3} \left(1 - \frac{Dt^2}{\gamma} \right) \exp(-Dt^2 / 2\gamma) . \end{aligned} \quad (97)$$

This shows that in the three-dimensional case, when $\beta = \sqrt{D/\gamma^2} \gg 1$ the correlation function decays to $\langle \mathbf{n}(t) \cdot \mathbf{n}(0) \rangle = 1/3$ on a rapid timescale, $\tau_1 = (\beta\gamma)^{-1}$, due to motions with different frequencies getting out of phase.

On longer timescales the value of $\boldsymbol{\omega}$ fluctuates, and the correlation function then decays to zero on a slower timescale describing the decay of correlations of $\boldsymbol{\omega}$. A precise understanding this limit requires us to carry out a more sophisticated analysis, as will be done in section 4.4 below. Before addressing this issue we consider how $\mathbf{n}(t)$ behaves when $\boldsymbol{\omega}$ varies slowly. We show that the angle between \mathbf{n} and $\boldsymbol{\omega}$ is an adiabatic invariant of the dynamics of (44). This shows that the decay of correlations of $\mathbf{n}(t)$ is governed by the diffusion of the direction of $\boldsymbol{\omega}$, implying that the timescale for the decay of correlations of $\mathbf{n}(t)$ is $O(\gamma^{-1})$ in the limit as $\beta \rightarrow \infty$.

To show that the angle θ between $\mathbf{n}(t)$ and $\boldsymbol{\omega}$ is an adiabatic invariant, consider the time evolution of

$$f \equiv \mathbf{n} \cdot \boldsymbol{\omega} = \omega \cos \theta \equiv \omega z \quad (98)$$

where $\omega = |\boldsymbol{\omega}|$ and where the second equality defines $z = \cos \theta$. From (44), the time derivative of f is

$$\frac{df}{dt} = \mathbf{n} \cdot \frac{d\boldsymbol{\omega}}{dt} = \left[\frac{z}{\omega} \boldsymbol{\omega} + \delta \mathbf{n}(t) \right] \cdot \frac{d\boldsymbol{\omega}}{dt} \quad (99)$$

where $\delta \mathbf{n}(t)$ oscillates on a timescale ω^{-1} about a mean value which is equal to zero when $\boldsymbol{\omega}$ is constant. When evaluating the drift of f , we neglect this rapidly oscillating term, and write

$$\left\langle \frac{df}{dt} \right\rangle \approx \frac{z}{\omega} \boldsymbol{\omega} \cdot \frac{d\boldsymbol{\omega}}{dt} = \frac{z}{2\omega} \frac{d\omega^2}{dt} . \quad (100)$$

Alternatively, from the definition $f = z\omega$, we find

$$\frac{df}{dt} = \omega \frac{dz}{dt} + z \frac{d\omega}{dt} = \omega \frac{dz}{dt} + \frac{z}{2\omega} \frac{d\omega^2}{dt} . \quad (101)$$

Comparing (100) and (101) we see that $\dot{z} = 0$, implying that θ is an adiabatic invariant, as stated above.

We have argued that in the limit as $\beta \rightarrow \infty$ the direction of $\mathbf{n}(t)$ is determined by the evolution of $\boldsymbol{\omega}$. This indicates that we can determine the rate of decay of $C(t)$ by determining the rate of decay of correlations of the angular momentum. The direction vector of the angular momentum, \mathbf{e}_ω , exhibits diffusion on the surface of the unit sphere with a diffusion coefficient \mathcal{D}_ω which we determine shortly. In a diffusion process, the expectation value of the spherical harmonic $\cos \theta$ decays exponentially: $\langle \cos \theta \rangle \sim \exp(-2\mathcal{D}_\omega t)$. We might therefore expect that when $\beta = \sqrt{D/\gamma^2} \gg 1$, the correlation function is well approximated by

$$\langle \mathbf{n}(t) \cdot \mathbf{n}(0) \rangle \sim \frac{1}{3} \exp(-\lambda t) + \frac{2}{3} \left(1 - \frac{Dt^2}{\gamma} \right) \exp(-Dt^2 / 2\gamma) . \quad (102)$$

and that $\lambda = 2\mathcal{D}_\omega$. This argument is, however, not satisfactory. When the deviation from equilibrium of the distribution of $\mathbf{n}(t)$ is well approximated by the spherical harmonic $\cos\theta$, the corresponding distribution of \mathbf{e}_ω might be a quite different combination of spherical harmonics. This question is most effectively addressed by the application of group theory to the Fokker-Planck equation, using results from sections 3.2 and 3.4 above.

We conclude this section by determining the diffusion coefficient for diffusion of the direction of $\boldsymbol{\omega}$. This direction is the unit vector $\mathbf{e}_\omega = \boldsymbol{\omega}/\omega$, which diffuses on the unit sphere with diffusion coefficient \mathcal{D}_ω . The change in the \mathbf{e}_ω in a short time δt is

$$\delta \mathbf{e}_\omega = \frac{1}{\omega^2} [\omega \delta \boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \delta \boldsymbol{\omega}) \boldsymbol{\omega}] \quad (103)$$

so that the expectation value of the square of the rotation angle is

$$\langle \delta \mathbf{e}_\omega^2 \rangle = \frac{2}{\omega^2} \langle \delta \omega_i^2 \rangle = \frac{4D\delta t}{\omega^2} . \quad (104)$$

The required diffusion coefficient is obtained by averaging this over the known distribution of $\boldsymbol{\omega}$:

$$\begin{aligned} \mathcal{D}_\omega &= \frac{\langle \delta \mathbf{e}_\omega^2 \rangle}{4\delta t} = D \int d\boldsymbol{\omega} \frac{P(\boldsymbol{\omega})}{\omega^2} \\ &= D \left(\frac{\gamma}{2\pi D} \right)^{3/2} \int_0^\infty d\omega \, 4\pi \omega^2 \frac{\exp(-\gamma\omega^2/2D)}{\omega^2} = \gamma . \end{aligned} \quad (105)$$

The correlation function for the direction of the angular momentum therefore decays as

$$\langle \mathbf{e}_\omega(t) \cdot \mathbf{e}_\omega(0) \rangle = \exp(-2\mathcal{D}_\omega t) = \exp(-2\gamma t) . \quad (106)$$

Numerical evidence indicates that the correlation function $C(t)$ decays at a different rate when $\beta \gg 1$: the decay rate in (102) is found to be $\lambda \approx 1.56\gamma$, rather than $\lambda = 2\gamma$. The difference is explained in the following section, 4.4.

4.4 Asymptotic solution of slowest mode

Here we use results derived from symmetry considerations (in sections 3.2 and 3.4) to identify the slowest decaying modes in the limit as $\beta \rightarrow \infty$. The objective is to determine solutions of (87) which decay exponentially in time, so that $\zeta_j(\omega, t) = \exp(-\lambda t) a_j(\omega)$. The functions $a_j(\omega)$ satisfy the eigenvalue equation

$$\begin{aligned} \hat{L}_0 a_0(\omega) + \sqrt{2}\beta\omega \mathcal{A}_{01} a_1(\omega) &= \lambda a_0(\omega) \\ \hat{L}_1 a_1(\omega) + \sqrt{2}\beta\omega [\mathcal{A}_{10} a_0(\omega) + \mathcal{A}_{12} a_2(\omega)] &= \lambda a_1(\omega) \\ \hat{L}_2 a_2(\omega) + \sqrt{2}\beta\omega \mathcal{A}_{21} a_1(\omega) &= \lambda a_2(\omega) \end{aligned} \quad (107)$$

The structure of the operator (88) implies that

$$\lim_{\omega \rightarrow 0} \frac{a_j(\omega)}{\omega^j} = C_j \quad (108)$$

for some constants C_j .

We are interested in the most slowly decaying solutions, which requires determining the eigenvalue λ with the largest real part. The structure of (107) suggests that the decay rates are expected to increase in proportion to β as $\beta \rightarrow \infty$. However, our discussion in section 4.3 indicates that there should be eigenfunctions which have a slow decay rate, $\lambda = O(\gamma)$ as $\beta \rightarrow \infty$. In order to identify these slow modes, note that the matrix \mathcal{A}_{ij} has a null eigenvector: this matrix is

$$\{\mathcal{A}_{ij}\} = \begin{pmatrix} 0 & \sqrt{\frac{2}{3}} & 0 \\ -\sqrt{\frac{2}{3}} & 0 & \sqrt{\frac{1}{3}} \\ 0 & -\sqrt{\frac{1}{3}} & 0 \end{pmatrix} \quad (109)$$

which has a null vector, $(1, 0, 2)$. The only way to obtain an eigenfunction with an eigenvalue which remains bounded as $\beta \rightarrow \infty$ is to assume that throughout most of the range of ω , we have

$$\mathcal{A}_{10} a_0(\omega) + \mathcal{A}_{12} a_2(\omega) \approx 0 . \quad (110)$$

Using this approximation, the equation for $a_2(\omega)$ can be written

$$\left[\hat{L}_2 + \rho \hat{L}_0 \right] a_2 = (1 + \rho) \lambda a_2 \quad (111)$$

where

$$\rho = \frac{\mathcal{A}_{21} \mathcal{A}_{12}}{\mathcal{A}_{01} \mathcal{A}_{10}} = \frac{1}{2} . \quad (112)$$

However, we have

$$\hat{L}_j = \hat{L}_0 + \frac{j(j+1)}{\omega^2} \quad (113)$$

so that $a_2(\omega)$ is an eigenfunction of

$$\hat{L}_j = \hat{L}_0 + \frac{6}{1 + \rho} \frac{1}{\omega^2} \quad (114)$$

with eigenvalue λ . This is an operator of the form (113) with an angular momentum quantum number j which satisfies $j(j+1) = 6/(1 + \rho) = 4$, that is,

$$j = \frac{\sqrt{17} - 1}{2} . \quad (115)$$

Thus it is argued that, in the limit as $\beta \rightarrow \infty$, there exist modes for which the eigenvalues are $O(\gamma)$, which satisfy a radial equation with an *irrational* value of the angular momentum, given by (115). It remains to identify the eigenvalues associated with this equation. The first step is to factor out the boundary condition at $\omega = 0$, writing

$$a_j(\omega) = \omega^j \Phi(\omega) . \quad (116)$$

Then it is useful to remove the Gaussian factor from the solution, writing $\Phi(\omega) = \exp(-\omega^2/2) \phi(\omega^2)$. The equation for $\phi(u)$ (with $u = \omega^2$) is:

$$u\phi'' + (j + 3/2 - u)\phi' + \frac{(\lambda - j)}{2}\phi = 0 . \quad (117)$$

Well behaved polynomial solutions exist only when $(\lambda - j) = 2n$ (n integer). Inserting the factor of γ which is required when we return to dimensioned equations, we conclude that the eigenvalue which gives the slowest rate of decay is

$$\lambda = -\gamma j = -\frac{\gamma(\sqrt{17} - 1)}{2} \quad (118)$$

and the full set of eigenvalues of the problem defined by equation (117) is

$$\lambda_n = -\gamma(j + 2n) , \quad n = 0, 1, 2, \dots . \quad (119)$$

The eigenfunctions can also be expressed in terms of generalised Laguerre polynomials, so that the Fokker-Planck equation may be regarded as exactly solvable in the limit as $\beta \rightarrow \infty$.

We have imposed two apparently incompatible conditions on the solutions $a_j(\omega)$, namely (108) and (110). We conclude this section by considering how these are reconciled. Consider the nature of the solutions $a_j(\omega)$ close to $\omega = 0$. Dimensionally, the problem is analogous to one with the following structure: $d^2\Phi/d\omega^2 + \beta\omega\Phi = 0$. The derivative terms becomes dominant when $\omega < \beta^{-1/3}$, so that the approximation (110) fails and (108) becomes applicable when $\omega\beta^{1/3} \ll 1$. The problem can be treated by applying standard asymptotic expansion methods [9]. Alternatively, one can determine the solutions of the problem by expanding the solution on a conveniently complete basis, and by truncation, reduce the solution of equations (107) to a matrix equation, as explained in Appendix A.

Figure 2 shows the numerically computed correlation functions for the three-dimensional Ornstein-Uhlenbeck process, compared with various asymptotic approximations. We also investigated the spectrum of the Fokker-Planck operator numerically. Using the eigenfunctions of the spherical harmonic

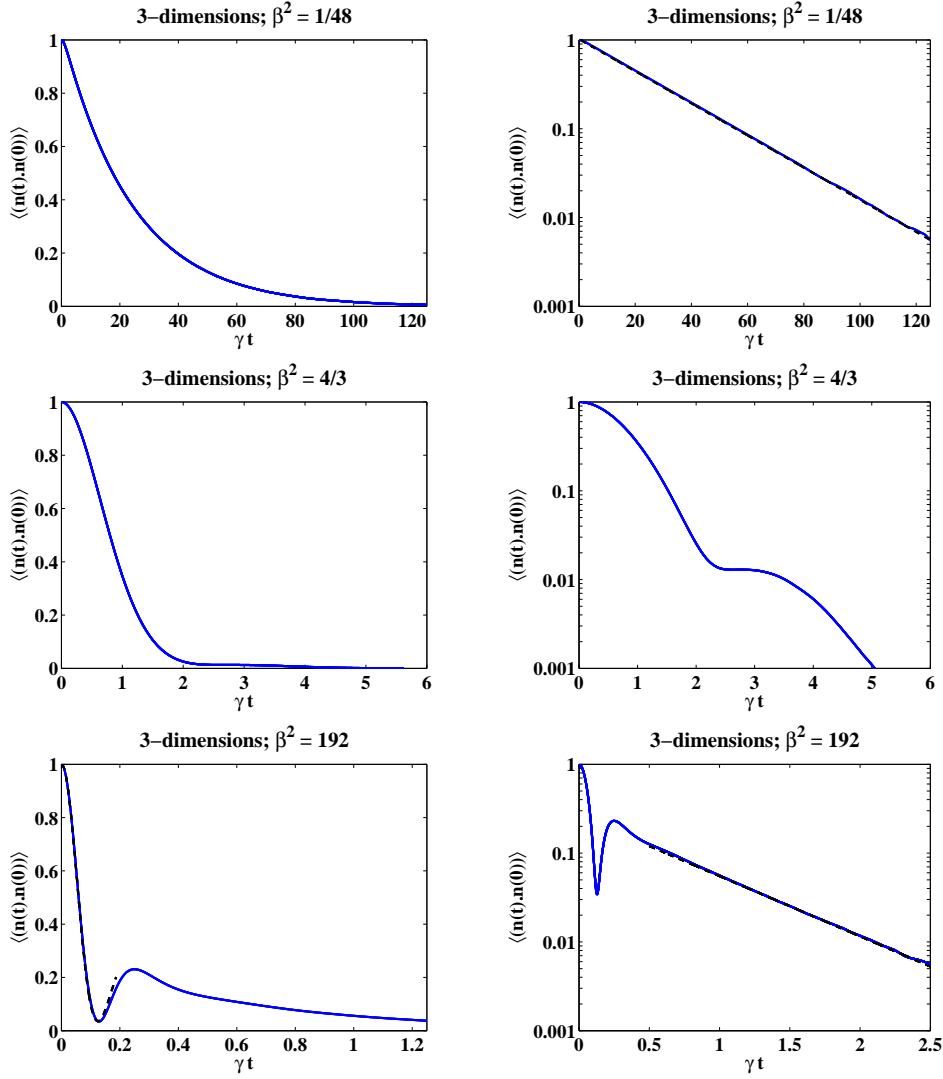


Fig. 2 Correlation function for the spherical Ornstein-Uhlenbeck process, for three values of $\beta^2 = D/\gamma^3$: $\beta^2 = 1/48$ (upper row), $\beta^2 = 4/3$ (middle row) and $\beta^2 = 192$ (lower row). The data for $\beta^2 = 1/48$ show good agreement with the exponential approximation to the correlation function, equation (92), which is applicable in the limit as $\beta \rightarrow 0$. The linear plot for $\beta^2 = 192$ shows good agreement with the transient described by equation (102), and the logarithmic plot demonstrates that the correlation function $C(t)$ behaves at long times as $C(t) \propto \exp(-(\sqrt{17} - 1)\gamma t/2)$, as predicted by equation (118).

oscillator as a basis set, this operator can be represented by an infinite-dimensional matrix. The formulae for the matrix elements are given in appendix A. We find that the spectrum of finite dimensional truncations converge as the size of the basis set increases, and we identify the converged eigenvalues with elements of the spectrum of the Fokker-Planck operator. Figures 3 and 4 illustrate the dependence of the eigenvalues upon β .

5 Applications to random tumbling

This paper has described a model for the statistics of a unit vector $\mathbf{n}(t)$ moving randomly but smoothly over the surface of a sphere. The model is a generalisation of the Ornstein-Uhlenbeck process to a spherical geometry. It has the merit of being susceptible to analytical treatments, including an exact

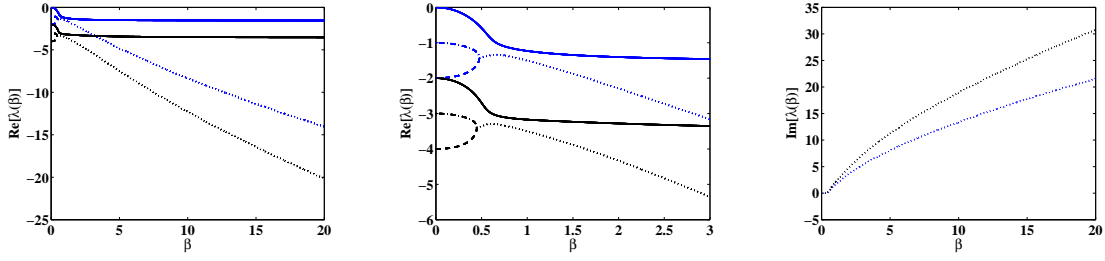


Fig. 3 The six slowest decaying eigenvalues from the spectrum of the Fokker-Planck operator of the spherical Ornstein-Uhlenbeck process as a function of β . Left: real part of the eigenvalues, showing the separation into levels which diverge as $\beta \rightarrow \infty$ and levels which approach a finite limit. Centre: real parts at greater magnification. At very small values of β , all eigenvalues are real. Pairs of branches collide at a finite value of β , giving rise to pairs of complex conjugate eigenvectors, whose real parts are shown by the dashed lines. Right: positive imaginary parts of the eigenvalues.

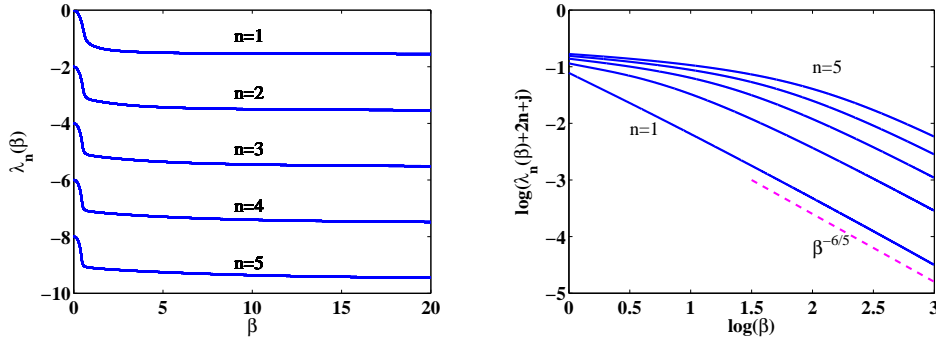


Fig. 4 The real-valued eigenvalues approach the spectrum $\lambda_n = j + 2n$ in the limit as $\beta \rightarrow \infty$, in accord with equation (118). The right-hand panel shows the convergence towards this limit as $\beta \rightarrow \infty$.

solution in two dimensions. It is of interest to consider the possible applications of this model, and the extent to which the model provides an accurate description of various systems.

The spherical Ornstein-Uhlenbeck model could be used to describe the tumbling of an object in a turbulent fluid flow. Examples include the rotational motion of rocks or dust grains in turbulent circumstellar discs [10], or of ice crystals in a convecting atmosphere [11]. The rotational motion of such bodies can influence their growth by aggregation or their evaporation by exposure to a source of radiant heat. Recently it has become possible to make detailed experimental studies of the orientation of a neutrally buoyant sphere in a turbulent fluid by matching images of an irregularly painted ball to photographs taken with the ball in a well defined orientation [12,13]. The direction $\mathbf{n}(t)$ of one axis through the sphere can be followed as a function of time. The model could also be used for the fluctuations of direction vector $\mathbf{n}(t)$ of a rod-like body in a turbulent fluid.

These remarks raise the question as to whether the spherical Ornstein-Uhlenbeck model will give an accurate description of the tumbling motion of a body. In the case of a small body in a turbulent flow, we argue below that the statistical properties of the velocity gradients of turbulence appear to make this a very good model. The orientation of a small body in a turbulent flow with velocity field $\mathbf{u}(\mathbf{r}(t), t)$ responds to the gradients of the velocity field, evaluated along the trajectory $\mathbf{r}(t)$ of the body (which can be assumed to be advected with the fluid). The velocity gradients form a matrix $\mathbf{A}(t)$, with elements $A_{ij}(t) = \partial u_i / \partial r_j(\mathbf{r}(t), t)$. It is convenient to write $\mathbf{A} = \mathbf{\Omega} + \mathbf{S}$, where $\mathbf{\Omega}$, the vorticity tensor, is antisymmetric and where \mathbf{S} , the strain-rate, is symmetric. The equation of motion for the direction vector $\mathbf{n}(t)$ of a microscopic ellipsoidal object in a fluid flow was obtained by Jeffery [14]. It can be written in the form

$$\frac{d\mathbf{n}}{dt} = \mathbf{\Omega}(t)\mathbf{n} + \frac{\alpha^2 - 1}{\alpha^2 + 1} [\mathbf{S}(t)\mathbf{n} - (\mathbf{n} \cdot \mathbf{S}(t)\mathbf{n})\mathbf{n}] \quad (120)$$

where α is the axis ratio of the ellipsoid. The same equation of motion applies to general axisymmetric bodies, provided they are small compared to any characteristic lengthscale of the flow [15], but the relation between α and the shape of the object is not known in general. In the case of a spherical particle or other object with $\alpha = 1$, the equation of motion (120) is of the same form as equation (44), if we interpret the vorticity $\boldsymbol{\omega}(t)$ as being the antisymmetric tensor corresponding to the axial vector $\boldsymbol{\omega}$, with elements related by $\omega_i = \epsilon_{ijk}\Omega_{jk}$.

Furthermore, the Lagrangian correlation function of the vorticity in turbulent flows has been investigated using simulations of turbulent flows by several authors [16, 17, 18]. The elements have mean value equal to zero and appear to be statistically independent. It was found that the correlation function of each elements can be fitted quite accurately by an exponential function:

$$\langle \Omega_{ij}(t)\Omega_{ij}(t') \rangle = \frac{D_v}{\gamma_v} \exp(-\gamma_v|t - t'|) . \quad (121)$$

The decay rate γ_v is of the order of the inverse of the Kolmogorov timescale τ_K of the turbulent flow, which is the shortest timescale of the fluid motion: $\tau_K = \sqrt{\nu/\mathcal{E}}$, where ν is the kinematic viscosity and where \mathcal{E} is the rate of dissipation per unit mass. Numerical evidence indicates that $\gamma_v \approx 1/(8.5\tau_K)$ at large Reynolds numbers [18]. The diffusion coefficient D_v can be related to γ_v by various kinematic constraints (discussed in [18]), giving $D_v = \gamma_v/12\tau_K^2$. These considerations suggest that the spherical Ornstein-Uhlenbeck model should describe tumbling of a small object in a turbulent flow, with a ‘universal’ value for the persistence angle

$$\beta_{\text{turb}} = \sqrt{\frac{D_v}{\gamma_v^3}} = \frac{1}{12\gamma_v^2\tau_K} \approx 2.4 . \quad (122)$$

However, we cannot conclude that correlation function $C(t) = \langle \mathbf{n}(t) \cdot \mathbf{n}(0) \rangle$ for objects in a turbulent fluid will correspond to that of our spherical Ornstein-Uhlenbeck model, because vorticity may have very different temporal variation, compared to the Ornstein-Uhlenbeck model, but still have the same correlation function. This point is illustrated by a calculation in appendix B, where we analyse a model for random motion on a circle, in which the angular velocity is determined by a telegraph process (examples of this type of model are considered in [19, 20]). The telegraph model has a correlation function of angular velocity which is an exponential function, equivalent to that of the circular Ornstein-Uhlenbeck process. However we show that the correlation function $\langle \mathbf{n}(t) \cdot \mathbf{n}(0) \rangle$ is very different for the two models. We conclude that the extent to which the spherical Ornstein-Uhlenbeck process is a good description of tumbling in a turbulent fluid must be tested by numerical simulations of turbulence. Our own numerical investigations on the tumbling of microscopic particles in turbulence indicate that the spherical Ornstein-Uhlenbeck model is not a good model for their correlation function [18]. This conclusion is consistent with studies by Shin and Koch [21], who presented data for $\langle \mathbf{n}(t) \cdot \mathbf{n}(0) \rangle$ in simulations of rod-like objects in fully developed driven turbulent flows.

It is only in cases where the statistics of the angular momentum fluctuations are a precise match to the spherical Ornstein-Uhlenbeck process that reliable predictions can be made about the correlation function defined by (6). In the case of an object tumbling in a very dilute gas, such as a small rock in the circumstellar disc of the star, the spherical Ornstein-Uhlenbeck model may be a very good description of the evolution of the angular momentum. Bombardment by microscopic dust grains can provide random impulses which change the angular momentum in the same way as the white-noise fluctuations in (45). And the damping due to motion in an extremely dilute gas is proportional to the relative velocity [22], consistent with the linear damping term in equation (45). We conclude that a small body tumbling in a very dilute gas is one example where the spherical Ornstein-Uhlenbeck model is an excellent description of a physical process.

6 Concluding remarks

Our study was motivated by recent works, which have characterized the orientation of particles transported by a turbulent flow [12, 13, 21, 23, 24]. These processes are naturally described by the random motion on a spherical surface. The work here has focused on arguably the simplest model for smooth

random motion on a sphere: the direction \mathbf{n} rotates with an angular velocity $\boldsymbol{\omega}$, which evolves according to an Ornstein-Uhlenbeck process.

The solution of this model depends on only one dimensionless parameter, which we refer to here as β , the persistence angle, which characterizes the rotation occurring during one correlation time of $\boldsymbol{\omega}$. We have characterised evolution of $\mathbf{n}(t)$ by analysing its correlation function $\langle \mathbf{n}(0)\mathbf{n}(t) \rangle$.

In two dimensions, we have obtained an explicit expression for the correlation function, in terms of very elementary functions. This was achieved by completely diagonalizing the Fokker-Planck operator and hence writing the correlation function as a series, whose sum can be explicitly determined.

In contrast, the motion on the three-dimensional sphere is more involved. This is largely due to the more complicated structure of the rotation group in three dimensions. Quite generally, the components of \mathbf{n} perpendicular to $\boldsymbol{\omega}$ rapidly rotate, hence decorrelate, whereas the component parallel to $\boldsymbol{\omega}$ remains unchanged, at least when $\boldsymbol{\omega}$ is constant. In the limit when the persistence angle is large, this leads to a two time-scale dynamics: a fast decorrelation of \mathbf{n} is observed, corresponding to the components of \mathbf{n} perpendicular to $\boldsymbol{\omega}$, followed by a much slower decorrelation of \mathbf{n} , corresponding to the component parallel to $\boldsymbol{\omega}$. Whereas the fast decorrelation can be understood quantitatively by using elementary considerations, the description of the decorrelation of \mathbf{n} at large times requires a determination of the largest eigenvalue of the Fokker-Planck operator. We have computed here the eigenvalues relevant to the long term evolution of $\langle \mathbf{n}(0) \cdot \mathbf{n}(t) \rangle$. Interestingly, in the large β limit, the problem reduces to a quantum harmonic oscillator with a irrational angular momentum.

The Ornstein-Uhlenbeck model is closely related to the equation describing the orientational degrees of freedom of small particles in turbulent flows [14,21], and numerical studies show that the vorticity of turbulent flows also has an exponential correlation [17,18]. However, we have observed here that caution should be observed in applying our model to rotation by turbulence. In two dimensions we showed that the correlation function of \mathbf{n} takes a very different form when the angular velocity is generated by a telegraph process, despite the fact that the correlation function of $\boldsymbol{\omega}$ are identical to our model.

In summary, the notion of the ‘persistence angle’ introduced here appears to be the most relevant parameter characterising random rotation. The Ornstein-Uhlenbeck model is the simplest description of random motion on a sphere, and it will surely find significant applications, beyond the example considered at the end of section 5. However, it is a poor model for rotations of small bodies driven by hydrodynamic turbulence [18].

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7 Appendix A: Matrix representation

7.1 Decomposition and projection.

The aim of this appendix is to project the system of partial differential equations (87) on the complete set of eigenfunctions of the spherical harmonic oscillator operator. The operators \hat{L}_l , introduced in equation (88) correspond to the radial part of the equation for the harmonic oscillator operators, equations (68, 69) associated with angular momentum quantum number $l = 0, 1$ and 2 respectively.

The eigenvalues of \hat{L}_l are thus $-(2n + l)$, n being the quantum number characterizing energy, and the corresponding radial eigenfunction being:

$$\phi_n^{(l)}(r) = \mathcal{N}_{nl} \exp(-r^2/2) r^l L_n^{(l+1/2)}(r^2) \quad (123)$$

where $L_n^{(\alpha)}$ is the generalized Laguerre polynomial [7], and the normalization constant is given by equation (107). The following discussion uses properties of the generalised Laguerre polynomials which are discussed in [25].

For each value of l , these eigenstates are orthogonal to each other, in the sense that:

$$(\phi_n^{(l)}, \phi_m^{(l)}) = \delta_{nm} . \quad (124)$$

One can therefore expand the functions $\psi^{(l)}$ in series of the $\phi_n^{(l)}$:

$$\psi^{(l)}(r) = \sum_{n=0}^{\infty} a_n^{(l)} \phi_n^{(l)}(r) . \quad (125)$$

This can then be inserted into the set of equations (107). Then, the equation corresponding to angular momentum l is projected on the set of modes $\phi_n^{(l)}(r)$. This leads to:

$$\begin{aligned} (\phi_n^{(0)}, \hat{L}_0 \psi^{(0)}) - 2/\sqrt{3} \beta(\phi_n^{(0)}, r \psi^{(1)}) &= \lambda(\phi_n^{(0)}, \psi^{(0)}) \\ (\phi_n^{(1)}, \hat{L}_1 \psi^{(1)}) + 2/\sqrt{3} \beta(\phi_n^{(1)}, r \psi^{(0)}) - \sqrt{2/3} \beta(\phi_n^{(1)}, r \psi^{(2)}) &= \lambda(\phi_n^{(1)}, \psi^{(1)}) \\ (\phi_n^{(2)}, \hat{L}_2 \psi^{(2)}) + \sqrt{2/3} \beta(\phi_n^{(2)}, r \psi^{(1)}) &= \lambda(\phi_n^{(2)}, \psi^{(2)}) \end{aligned} \quad (126)$$

Clearly,

$$(\phi_n^{(l)}, \hat{L}_l \psi^{(l)}) = -a_n^{(l)} (2n + l) . \quad (127)$$

The scalar products of $(\phi_n^{(0)}, r \psi^{(1)})$ and $(\phi_n^{(1)}, r \psi^{(0)})$, on one hand, and $(\phi_n^{(1)}, r \psi^{(2)})$ and $(\phi_n^{(2)}, r \psi^{(1)})$ on the other hand involve the calculations of the two integrals:

$$\begin{aligned} I_{n,m}^{0,1} &\equiv \int_0^{\infty} \exp(-r^2) r^{0+1} \times r \times r^2 L_n^{(1/2)}(r^2) L_m^{(3/2)}(r^2) \, dr \\ &= \frac{1}{2} \int_0^{\infty} \exp(-u) u^{3/2} L_n^{(1/2)}(u) L_m^{(3/2)}(u) \, du \end{aligned} \quad (128)$$

and:

$$\begin{aligned} I_{n,m}^{1,2} &\equiv \int_0^{\infty} \exp(-r^2) r^{1+2} \times r \times r^2 L_n^{(3/2)}(r^2) L_m^{(5/2)}(r^2) \, dr \\ &= \frac{1}{2} \int_0^{\infty} \exp(-u) u^{5/2} L_n^{(3/2)}(u) L_m^{(5/2)}(u) \, du . \end{aligned} \quad (129)$$

These two integrals can be easily computed, by using the following relation between generalized Laguerre polynomials:

$$L_n^{(\alpha)}(x) = L_n^{(\alpha+1)}(x) - L_{n-1}^{(\alpha+1)}(x) \quad (130)$$

and the normalization integral (73). One thus finds:

$$\begin{aligned} I_{n,m}^{0,1} &= \frac{1}{2} \int_0^{\infty} \exp(-u) u^{3/2} L_m^{(3/2)}(u) (L_n^{(3/2)}(u) - L_{n-1}^{(3/2)}(u)) \, du \\ &= (\delta_{n,m} - \delta_{n-1,m}) \frac{\Gamma(5/2 + m)}{2 m!} \end{aligned} \quad (131)$$

and

$$\begin{aligned} I_{n,m}^{1,2} &= \frac{1}{2} \int_0^{\infty} \exp(-u) u^{5/2} L_m^{(5/2)}(u) (L_n^{(5/2)}(u) - L_{n-1}^{(5/2)}(u)) \, du \\ &= (\delta_{n,m} - \delta_{n-1,m}) \frac{\Gamma(7/2 + m)}{2 m!} . \end{aligned} \quad (132)$$

Using these results, the set of equations (126) reduces to a matrix equation, with a relatively simple (band-) structure, as we explain below.

7.2 Matrix equations

It is now a simple matter to rewrite the matrix equations for the quantities $a_n^{(l)}$, defined in equation (125). Specifically, from Eq. (126), one obtains the system of equations:

$$\begin{aligned} -2na_n^{(0)} - 2/\sqrt{3}\beta \sum_{m=0}^{\infty} A(n, m) a_m^{(1)} &= \lambda a_n^{(0)} \\ -(2n+1)a_n^{(1)} + 2\sqrt{3}\beta \sum_{m=0}^{\infty} B(n, m) a_m^{(0)} - \sqrt{2/3}\beta \sum_{m=0}^{\infty} C(n, m) a_m^{(2)} &= \lambda a_n^{(1)} \\ -(2n+2)a_n^{(2)} + \sqrt{2/3}\beta \sum_{m=0}^{\infty} D(n, m) a_m^{(1)} &= \lambda a_n^{(2)} \end{aligned} \quad (133)$$

where

$$\begin{aligned} A(n, m) &= \frac{I_{n,m}^{0,1}}{\mathcal{N}_{n0}\mathcal{N}_{m1}} & B(n, m) &= \frac{I_{m,n}^{0,1}}{\mathcal{N}_{m0}\mathcal{N}_{n1}} \\ C(n, m) &= \frac{I_{n,m}^{1,2}}{\mathcal{N}_{n1}\mathcal{N}_{m2}} & D(n, m) &= \frac{I_{m,n}^{1,2}}{\mathcal{N}_{m1}\mathcal{N}_{n2}}. \end{aligned} \quad (134)$$

In fact, in view of the structure of the scalar products Eq. (131,132), the matrix equations Eq. (133) contain in fact very few terms. Explicitly,

$$\begin{aligned} -2na_n^{(0)} - 2\beta/\sqrt{3} \left[A(n, n) a_n^{(1)} - A(n, n-1) a_{n-1}^{(1)} \right] &= \lambda a_n^{(0)} \\ -(2n+1)a_n^{(1)} + \beta/\sqrt{3} \left[2 \left(B(n, n) a_n^{(0)} - B(n, n+1) a_{n+1}^{(0)} \right) \right. \\ &\quad \left. - \left(C(n, n) a_n^{(2)} - C(n, n-1) a_{n-1}^{(2)} \right) \right] &= \lambda a_n^{(1)} \\ -(2n+2)a_n^{(2)} + \sqrt{2/3}\beta \left[D(n, n) a_n^{(1)} - D(n, n+1) a_{n+1}^{(1)} \right] &= \lambda a_n^{(2)}. \end{aligned} \quad (135)$$

The structure of the matrix defined by equations (135), is easy to program in a routine that can diagonalize a real matrix, with the infinite-dimensional matrix truncated to a finite size by including only coefficients $a_n^{(l)}$ with $n \leq n_{\max}$. Although the structure of the matrix is very sparse – the matrix a simple band structure – a general routine from NAG has been used to determine the spectrum. We checked that the eigenvalues converge as the number of coefficients n_{\max} increases. We were able to obtain the eigenvalues with the largest real parts, i.e., the one that correspond to the slowest decay of the correlation function.

8 Appendix B: Telegraph-noise model

When solving physical problems it is often tacitly assumed that the correlation function of a stochastic signal is sufficient to characterise its properties. For example, if we were to use a different stochastic process to generate the angular velocity, we might expect that the correlation function $C(t) = \langle \mathbf{n}(t) \cdot \mathbf{n}(0) \rangle$ would be little changed if the correlation function of the angular velocity remains the same, namely $\langle \omega(t)\omega(0) \rangle = (D/\gamma) \exp(-\gamma|t|)$. This assumption may not be valid in general. To illustrate this point, here we determine the correlation function (6) for random motion on a circle in the case where the angular velocity is generated by a telegraph noise process, which has precisely the same exponential correlation function as the Ornstein-Uhlenbeck process. The correlation function of the direction vector, defined by (6), is found to be very different from (35).

In the telegraph noise model the angular velocity $\omega(t)$ takes just two discrete values, $\pm\omega_0$ (where ω_0 is a constant). The angular momentum makes random transitions between these values, with a rate constant R (so that the probability of transition in a short time interval of length δt is $\delta P = R\delta t$).

The rotation angle θ satisfies $\dot{\theta} = \omega(t)$, as before. Define $P_{\pm}(\theta, t)$ to be the probability density to be located at θ at time t , with $\omega(t) = \pm\omega_0$. The probability densities satisfy

$$\frac{\partial P_{\pm}}{\partial t} = -\pm\omega_0 \frac{\partial P_{\pm}}{\partial \theta} + R P_{\mp} - R P_{\pm} . \quad (136)$$

Formally, this equation can be written $\partial_t |P\rangle = \hat{\mathcal{F}} |P\rangle$, where $|P\rangle$ is a function vector representing $(P_+(\theta, t), P_-(\theta, t))$. This equation can be solved by seeking eigenfunctions of $\hat{\mathcal{F}}$ in the form $|\psi\rangle = \exp(in\theta)(c_+, c_-)$. The vector $\mathbf{c} = (c_+, c_-)$ is an eigenvector of the 2×2 matrix

$$\mathbf{F} = \begin{pmatrix} -in\omega_0 - R & R \\ R & in\omega_0 - R \end{pmatrix} . \quad (137)$$

This matrix has eigenvalues

$$\lambda_{n\pm} = -R \pm \sqrt{R^2 - n^2\omega_0^2} . \quad (138)$$

The general solution of (136) can be expressed as a linear combination of eigenfunctions:

$$\begin{pmatrix} P_+(\theta, t) \\ P_-(\theta, t) \end{pmatrix} = \sum_{n=-\infty}^{\infty} a_{n+} \exp(\lambda_{n+}t + in\theta) \begin{pmatrix} R \\ in\omega_0 + \sqrt{R^2 - n^2\omega_0^2} \end{pmatrix} \\ + a_{n-} \exp(\lambda_{n-}t + in\theta) \begin{pmatrix} R \\ in\omega_0 - \sqrt{R^2 - n^2\omega_0^2} \end{pmatrix} . \quad (139)$$

In order to evaluate the correlation function (6) we must compute $C(t) = \langle \cos \theta \rangle$, with the initial condition $P_{\pm}(\theta) = \delta(\theta)/2$. Thus

$$\begin{aligned} C(t) &= \int_0^{2\pi} d\theta \cos \theta [P_+(\theta, t) + P_-(\theta, t)] \\ &= 2\pi \text{Re} \left[a_{1+} \left(R + i\omega_0 + \sqrt{R^2 - \omega_0^2} \right) \exp(\lambda_{1+}t) + a_{1-} \left(R + i\omega_0 - \sqrt{R^2 - \omega_0^2} \right) \exp(\lambda_{1-}t) \right] . \end{aligned} \quad (140)$$

The coefficients $a_{n\pm}$ are easily determined from the initial distribution:

$$a_{n\pm} = \frac{\sqrt{R^2 - n^2\omega_0^2} \pm (R - in\omega_0)}{8\pi R \sqrt{R^2 - n^2\omega_0^2}} \quad (141)$$

and hence

$$C(t) = \frac{1}{2} [\exp(\lambda_{1+}t) + \exp(\lambda_{1-}t)] + \frac{R}{2\sqrt{R^2 - \omega_0^2}} [\exp(\lambda_{1+}t) - \exp(\lambda_{1-}t)] \quad (142)$$

where $\lambda_{\pm} = \lambda_{1\pm} = -R \pm \sqrt{R^2 - \omega_0^2}$. A similar and somewhat simpler calculation gives the correlation function of $\omega(t)$:

$$\langle \omega(t)\omega(0) \rangle = \omega_0^2 \exp(-2R|t|) . \quad (143)$$

Because this correlation function has the same structure as that of the Ornstein-Uhlenbeck process, we can define the parameters ω_0 , R of the telegraph noise model in terms of the parameters γ , D of the Ornstein-Uhlenbeck process: by comparison of (143) with (3) we have

$$\gamma = 2R , \quad D = 2R\omega_0^2 , \quad \beta = \frac{\omega_0}{2R} . \quad (144)$$

Expressed in terms of the same variables as the Ornstein-Uhlenbeck process, the correlation function of the telegraph noise model is

$$C(t) = \frac{\sqrt{1 - 4\beta^2} + 1}{2\sqrt{1 - 4\beta^2}} \exp \left[-\frac{1 - \sqrt{1 - 4\beta^2}}{2} \gamma t \right] + \frac{\sqrt{1 - 4\beta^2} - 1}{2\sqrt{1 - 4\beta^2}} \exp \left[-\frac{1 + \sqrt{1 - 4\beta^2}}{2} \gamma t \right] . \quad (145)$$

This correlation function is significantly different from (35); for example (145) is oscillatory when $\beta > 1/2$.

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